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We show that bipartite entanglements involving non-orthogonal states are *necessarily nonmaximally* entangled, however *small* the non-orthogonality may be! How the deviation from maximal entanglement is related to nonorthogonality is quantified by using two independent measures of entanglement corresponding to the violation of Bell's inequality and the entropy measure respectively. This result is true even if one of the subsystems has orthogonal states. An application of this is that a maximal violation of Bell's inequality in entangled neutral kaons is not possible in the presence of CP violation.

I. INTRODUCTION

Quantum entanglement is usually studied using entangled orthogonal states. Although entangled non-orthogonal states play important role in quantum cryptography [1], they have received less attention compared to their orthogonal counterparts, such as in connection with Bell's inequalities (BI) [2]. Examples of entangled non-orthogonal states are readily available; entangled coherent states fall under this category and experiments on quantum teleportation involve them [3,4]. A Schrödinger cat state which has been developed for $SU(2)$ coherent states could be extended to an entanglement of nonorthogonal $SU(2)$ coherent states [5]. The entangled neutral kaons in the presence of CP violation involve such nonorthogonal entangled states [6]. Hence a study of the question of how non-orthogonality is related to maximal entanglement is relevant to the questions concerning teleportation, dense-coding and entanglement swapping involving entangled non-orthogonal states. In particular, we may note that perfect teleportation is possible essentially for maximally entangled states [7] while arguments for nonlocality such as those of the Hardy-type [8] hold for nonmaximally entangled states.

In this paper it is demonstrated that bipartite entangled systems having non-orthogonal states are *necessarily non-maximally* entangled however *small* the non-orthogonality may be, which is true even if one of the subsystems has orthogonal states. A quantitative relation between the departure from maximal entanglement and nonorthogonality is derived by using two independent measures of entanglement corresponding to the amount of violation of BI and the entropy measure respectively. An application of this result is discussed for entangled neutral kaons in the presence of CP violation. We also point out directions for some further studies.

II. PRELIMINARIES

A bipartite entangled state can, in general, be written as

$$|\Psi>^{AB} = \mu|\alpha>^A|\beta>^B + \nu|\gamma>^A|\delta>^B, \tag{1}$$

where $|\alpha>^A$ and $|\gamma>^A$ are the states of system 1 and similarly for $|\beta>^B$ and $|\delta>^B$ for system 2 with complex coefficients μ and ν. A bipartite entangled state involving nonorthogonal states would have the property that the overlaps $^A<\alpha|\gamma>^A$ and $^B<\delta|\beta>^B$ are non-zero. The two linearly independent nonorthogonal states $|\alpha>^A$ and $|\gamma>^A$ that span a two-dimensional subspace of the Hilbert space can be chosen such that

$$|\alpha>^A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^A, \ |\gamma>^A = \begin{pmatrix} \mathcal{N}^A \\ y \end{pmatrix}^A \tag{2}$$

where $\mathcal{N}^A = \sqrt{1-|y|^2}$ where $y = ^A<\alpha|\gamma>^A$. Similarly for system B one has

$$|\delta>^B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^B, \ |\beta>^B = \begin{pmatrix} \mathcal{N}^B \\ x \end{pmatrix}^B \tag{3}$$

where $\mathcal{N}^B = \sqrt{1-|x|^2}$ where $x = ^B<\delta|\beta>^B$. In this basis the state (1) can be written as

$$\Psi^{AB} = \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}^A \otimes \begin{pmatrix} \mathcal{N}^B \\ x \end{pmatrix}^B + \nu \tag{4}$$

$$\begin{pmatrix} \mathcal{N}^A \\ y \end{pmatrix}^A \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^B = \begin{pmatrix} 0 \\ \nu\mathcal{N}^A \\ \mu\mathcal{N}^B \\ \mathcal{M} \end{pmatrix},$$

where $\mathcal{M} \equiv \mu x + \nu y$ and the superscripts A and B are ignored for cases where no ambiguity arises. Also the normalization of (4) requires that

$$|\mu\mathcal{N}^B|^2 + |\nu\mathcal{N}^A|^2 + |\mathcal{M}|^2 = 1. \tag{5}$$

The state (4) is a pure state with density matrix $\rho^{AB} = \Psi^{AB}\Psi^{AB\dagger}$ and the reduced density matrices ρ^A and ρ^B for systems A and B are

$$\rho^A = \mathrm{Tr}_B\rho^{AB} = \begin{pmatrix} |\nu\mathcal{N}^A|^2 & \nu\mathcal{N}^A\mathcal{M}^* \\ \nu^*\mathcal{N}^A\mathcal{M} & |\mu\mathcal{N}^B|^2 + |\mathcal{M}|^2 \end{pmatrix}$$

$$\rho^B = \mathrm{Tr}_A\rho^{AB} = \begin{pmatrix} |\nu\mathcal{N}^B|^2 & \nu\mathcal{N}^B\mathcal{M}^* \\ \nu^*\mathcal{N}^B\mathcal{M} & |\mu\mathcal{N}^A|^2 + |\mathcal{M}|^2 \end{pmatrix} \tag{6}$$

which satisfy

$$det\ \rho^A = |\mu\nu\mathcal{N}^B\mathcal{N}^A|^2 = det\ \rho^B. \tag{7}$$

Both ρ^A and ρ^B have identical eigenvalues given by

$$\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4|\mu\nu\mathcal{N}^B\mathcal{N}^A|^2} \tag{8}$$

with corresponding eigenvectors $|\pm>^A$ and $|\pm>^B$ for ρ^A and ρ^B respectively. Using the general theory of Schmidt decomposition the state (4) can be expressed as

$$|\Psi>^{AB} = c_-|->^A\,|->^B + c_+|+>^A\,|+>^B, \tag{9}$$

with

$$|c_\pm|^2 = \lambda_\pm,\ \ |c_-|^2 + |c_+|^2 = 1, \tag{10}$$

for studying the violation of BI [3]. Considering each two-state system as a spin-$\frac{1}{2}$ system, the Hermitian operators $\hat{\Theta}$ for each system A, B are chosen to have the general form

$$\begin{aligned}\hat{\Theta} = {} & \cos\chi[\,|+><+| - |-><-|\,] + \\ & \sin\chi[\,e^{i\phi}|+><-| + e^{-i\phi}|-><+|\,],\end{aligned} \tag{11}$$

that corresponds to the components of a 'spin' operator along the axis determined by the angles χ and ϕ. For the choices

$$\begin{aligned}\chi^A &= 0,\ \ \chi^{'A} = \pi/2 \\ \chi^B &= -\chi^{'B} = \cos^{-1}\,[1 + |2c_+c_-|^2]^{-\frac{1}{2}} \\ \phi^A + \phi^B &= \phi^{'A} + \phi^{'B} = \phi_+ - \phi_-\end{aligned} \tag{12}$$

where $\phi_\pm$ are the phases of $c_\pm$, the expectation value of the Bell operator

$$\hat{B} = \hat{\Theta}^A\hat{\Theta}^B + \hat{\Theta}^A\hat{\Theta}^{'B} + \hat{\Theta}^{'A}\hat{\Theta}^B - \hat{\Theta}^{'A}\hat{\Theta}^{'B} \tag{13}$$

for the state (9) can be shown to be

$$B \equiv <\Psi|\hat{B}|\Psi> = 2\sqrt{1 + |2c_+c_-|^2} > 2. \tag{14}$$

It was pointed out by Mann et. al. [3] that a violation of BI always occur. But they did not investigate the question concerning maximal violation of BI for non-orthogonal entanglements.

III. THE VIOLATION OF BI AND NONORTHOGONAL STATES

A. Maximal Violation of BI

In this subsection we discuss whether or not a maximal violation of BI is possible when both or one of the pairs of entangled states is mutually nonorthogonal and the above mentioned Bell operator is used. We first note from eqn.(14) that the degree of violation of BI depends on the values of $c_{\pm}$. It is evident that maximal violation occurs when

$$|c_+ c_-|^2 = \frac{1}{4} \tag{15}$$

Solving the above equation along with the normalization condition $|c_+|^2 + |c_-|^2 = 1$ one obtains $|c_+|^2 = |c_-|^2 = \frac{1}{2}$. Substitution of this in the expression for the eigenvalues of the reduced density matrices yields

$$(1-|x|^2)(1-|y|^2)|\mu\nu|^2 = \frac{1}{4} \tag{16}$$

The normalization condition given by eqn.(5) can be written as

$$|\mu|^2 + |\nu|^2 + 2|\mu||\nu||x||y|\cos\eta = 0, \tag{17}$$

with $\eta = (\theta_1 - \theta_3 + \theta_2 - \theta_4)$ where $\theta_j, (j = 1, 2, 3, 4)$ correspond to the phase of μ, x, ν, y respectively. We will show below that for the cases involving nonorthogonal bases there cannot be a maximal violation of BI because eqn.(16) and eqn.(17) cannot be simultaneously solved.

Case I: *Nonorthogonal-Nonorthogonal(NN) Case*

Substitution of eqn.(16) into eqn.(17) yields in this case

$$aq^2 + bq + c = 0, \tag{18}$$

where $q = |\mu|^2$ and the coefficients a, b, c are given by

$$\begin{aligned} a &= 4(1-|x|^2)(1-|y|^2) \\ b &= a\Big[\sqrt{\frac{|x|^2|y|^2}{(1-|x|^2)(1-|y|^2)}}\cos\eta - 1\Big] \\ c &= 1. \end{aligned} \tag{19}$$

The positive definiteness of q requires $b^2 - 4ac \geq b^2$ when $b > 0$ and $b^2 - 4ac \leq b^2$ when $b < 0$. The first of these conditions cannot be satisfied for the given a, b and c. But the second condition is satisfied when

$$\cos\eta \leq \frac{\sqrt{(1-|x|^2)(1-|y|^2)}}{|x||y|}. \tag{20}$$

Since q has to be real one has $b^2 \geq 4ac$ which in turn implies that

$$\cos\eta \geq \frac{1+\sqrt{(1-|x|^2)(1-|y|^2)}}{|x||y|}, \tag{21}$$

which is clearly in conflict with eqn.(20). Since $-1 \leq \cos\eta \leq 1$ the above condition requires

$$\frac{1+\sqrt{(1-|x|^2)(1-|y|^2)}}{|x||y|} \leq -1 \tag{22}$$

which cannot be satisfied. Hence q cannot be real which means that simultaneous solution of eqn.(16) and eqn.(17) is not possible. Thus it is not possible to have maximal violation of BI in this case.

Case II:*Orthogonal-Nonorthogonal(ON) Case*

In this case, bases for the system A is chosen to be orthogonal which means $|y| = 0$ and $\mathcal{N}^A = 1$. The coefficients of the quadratic equation involving q are given by

$$a = 4(1-|x|^2);\ b = -a;\ c = 1. \tag{23}$$

Although the positive definiteness of q may be satisfied in this case, q cannot be real. Because for q to be real one should have $(1-|x|^2) \geq 1$ which is not possible. Hence we find that even if one of the bases is orthogonal still then it is not possible to have maximal violation of BI in an entangled bipartite system.

Case III: *Orthogonal-Orthogonal(OO) Case*

The orthogonality for both bases requires $|y| = 0$ and $|x| = 0$. The coefficients of the quadratic equation involving q are given by

$$a = 4;\ b = -a;\ c = 1. \tag{24}$$

It is evident that there is a maximal violation of BI in this case because $q = \frac{1}{2}$ is a solution which in turn means $|\mu|^2 = |\nu|^2 = \frac{1}{2}$.

B. Non-Maximal Violation of BI

Now we discuss the explicit relationship between the parameters x and y (that represent non-orthogonality of the bases) and the parameter d that denotes the deviation from the maximal violation of BI. We define the parameter d by the following equation:

$$\sqrt{1+|2c_+c_-|^2} = \sqrt{2-d} \tag{25}$$

where $0 \leq d \leq 1$ and the lower limit corresponds to maximal violation of BI. In this case eqn.(16) becomes a function of d as follows:

$$(1-|x|^2)(1-|y|^2)|\mu\nu|^2 = \frac{1-d}{4} \tag{26}$$

Substituting the above eqn. into eqn. (17) yields the quadratic equation for the *NN* case

$$q^2 + bq + c = 0, \tag{27}$$

where $q = |\mu|^2$ and the coefficients b, c are given by

$$\begin{aligned} b &= [\sqrt{\frac{|x|^2|y|^2(1-d)}{(1-|x|^2)(1-|y|^2)}}\cos\eta - 1] \\ c &= \frac{(1-d)}{4(1-|x|^2)(1-|y|^2)}. \end{aligned} \tag{28}$$

The most general solution to eqn.(27) is

$$\begin{aligned} |\mu|^2 = \frac{1}{2}[&(1 - X\cos\eta) \\ &\pm\sqrt{(1-X\cos\eta)^2 - \frac{X^2}{|x|^2|y|^2}}], \end{aligned} \tag{29}$$

where

$$X = \sqrt{\frac{|x|^2|y|^2(1-d)}{(1-|x|^2)(1-|y|^2)}} \geq 0. \tag{30}$$

This solution reduces to maximal violation of BI in various cases discussed earlier for the case $d = 0$. Similar relations for the *ON* case and *OO* case respectively are given by

$$\begin{aligned} |\mu|^2 &= \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{d-|x|^2}{1-|x|^2}} \\ |\mu|^2 &= \frac{1}{2} \pm \frac{1}{2}\sqrt{d}. \end{aligned} \tag{31}$$

It is easy to see from the above expressions that there is no maximal violation of BI in ON case whereas there is a maximal violation of BI in OO case. Since the maximal violation of BI provides a measure of the degree of entanglement we can now indicate explicitly how the degree of entanglement (denoted by the deviation parameter d) changes with the overlap of the nonorthogonal bases:

$$
\begin{aligned}
d = 1 - 4(1-|x|^2)(1-|y|^2)[|\mu|^2 + \\
|\mu|^4(2|x|^2|y|^2\cos^2\eta - 1) \pm \sqrt{2}Z|\mu|^3|x||y|\cos\eta],
\end{aligned}
\tag{32}
$$

where

$$
Z = \sqrt{2(1-|\mu|^2) + |\mu|^2|x|^2|y|^2(1+\cos 2\eta)}. \tag{33}
$$

The above relation reduces to

$$
\begin{aligned}
d &= 1 - 4|\mu|^2(1-|\mu|^2)(1-|y|^2) \\
d &= 1 - 4|\mu|^2(1-|\mu|^2).
\end{aligned}
\tag{34}
$$

for the ON case and OO case respectively. It is evident from the above relations that for the OO case d vanishes, thereby implying a maximal violation of BI.

IV. MEASURE OF ENTANGLEMENT AND NONORTHOGONAL STATES

Another way of seeing how the nonorthogonality of the bases can affect the measure of entanglement is to note that the von Neumann entropy measure of entanglement of bipartite state Ψ^{AB} is given by

$$
\mathcal{E}(\psi^{AB}) = -\mathrm{Tr}(\rho^A \ln \rho^A) = -\mathrm{Tr}(\rho^B \ln \rho^B), \tag{35}
$$

where the reduced density matrices ρ^A and ρ^B are given as a function of the parameters x and y through eqn.(6). The above expression can be written as [9]

$$
\begin{aligned}
\mathcal{E}(\psi^{AB}) &= h(\frac{1+\sqrt{1-C^2}}{2}); \\
h(Z) &= -Z\log_2 Z - (1-Z)\log_2(1-Z),
\end{aligned}
\tag{36}
$$

where the *concurrence* C is defined as

$$
C(\Psi) = |<\Psi|\tilde{\Psi}>| \tag{37}
$$

The $|\tilde{\Psi}>$ that appears in the definition of C is given by

$$
|\tilde{\Psi}> = \sigma_y|\Psi^*>, \tag{38}
$$

where $|\Psi^*>$ is the complex conjugate of $|\Psi>$ when it is expressed in a fixed basis such as $\{|\uparrow>, |\downarrow>\}$, and σ_y expressed in that same basis is the usual Pauli matrix. For a spin-$\frac{1}{2}$ particle this is the standard time reversal operation and it reverses the direction of the spin.

In our case, we have

$$
\begin{aligned}
C^2(\Psi^{AB}) &= 4\det\rho^A = 4\det\rho^B \\
&= 4(1-|x|^2)(1-|y|^2)|\mu\nu|^2.
\end{aligned}
\tag{39}
$$

It is easy to see from the eqn.(39) and eqn. (16) that the *concurrence* C becomes 1 for the maximally entangled states. For any non-maximally entangled state it is always less than 1. Thus eqn.(39) and eqn. (36) indicate how the entropy measure of the entanglement depends on the nonorthogonality of the bases.

V. APPLICATIONS

A. Neutral Kaons System

As an application of the formalism developed above we show here the relationship between the deviation from maximal violation of BI (parameter d) and the CP violation parameter ϵ for neutral kaons. For them the mass eigenstates $|K_S>$ and $|K_L>$ are written in terms of CP eigenstates $|K_1>$ and $|K_2>$ as

$$\begin{aligned} |K_S> &= \frac{1}{\sqrt{1+|\epsilon|^2}}[|K_1>+\epsilon|K_2>] \\ |K_L> &= \frac{1}{\sqrt{1+|\epsilon|^2}}[|K_2>+\epsilon|K_1>], \end{aligned} \tag{40}$$

where

$$|K_{1,2}>=\frac{1}{\sqrt{2}}[|K^0>\pm|\bar{K^0}>] \tag{41}$$

In this system, the nonorthogonal overlaps are given by

$$|x|^2=|y|^2=|<K_S|K_L>|^2=(\frac{Re\epsilon}{1+|\epsilon|^2})^2 \tag{42}$$

Taking into account the charge conjugation ($=-1$) of the Φ–meson the EPR-Bohm type entangled state of the neutral kaon pair coming from Φ decay can be written as

$$|\Phi\rangle=\frac{1}{\sqrt{2}}\left[|K^0\rangle\otimes|\bar{K^0}\rangle-|\bar{K^0}\rangle\otimes|K^0\rangle\right]. \tag{43}$$

The neutral kaons then fly apart getting spatially separated. Their time evolution under weak interactions is given by

$$|\Phi\rangle=\frac{N(t)}{\sqrt{2}}\left[|K_S\rangle\otimes|K_L\rangle-|K_L\rangle\otimes|K_S\rangle\right], \tag{44}$$

where $|N(t)|=(1+|\epsilon|^2)/(|1-\epsilon^2|)\times e^{-\frac{1}{2}(\Gamma_S+\Gamma_L)t}$ reflecting the extinction of the beams via weak interaction induced kaon decays but without modifying the perfect antisymmetry of the initial state.

For this system the deviation parameter d is given as a function of ϵ as follows:

$$d=1-\{1-(\frac{Re\epsilon}{1+|\epsilon|^2})^2\}^2[1+\sqrt{2}Y\cos\eta(Re\epsilon)^2(1+|\epsilon|^2)^{-4}], \tag{45}$$

where

$$Y=\sqrt{2}\cos\eta(Re\epsilon)^2\pm\sqrt{(Re\epsilon)^4+(Re\epsilon)^4\cos 2\eta+2(1+|\epsilon|^2)^4}. \tag{46}$$

Hence from eqn.(29) it is clear that there cannot be a maximal violation of BI for entangled neutral kaons in the presence of CP violation. But the deviation d can be made quite small compared to 1 by suitable choice of η since CP violation parameter $\epsilon\approx 10^{-3}$.

VI. CONCLUSION

To summarise, we have shown that the bipartite entangled systems involving non-orthogonal states are *non-maximally* entangled, irrespective of how *small* the non-orthogonality is. Now, note that the physical content of entanglement (e. g., the possibility of perfect teleportation, maximal violation of local realism) is contingent on the degree of entanglement and in particular on whether entanglement is maximal. However, any measure of entanglement changes under a *non-unitary* transformation that connects a non-orthogonal basis with an orthogonal basis. Thus our work implies that, no matter how small this non-unitarity is, such a transformation *changes* the physical content of entanglement. Implications of this feature could be interesting for further studies.

One application of the above result is pointed out by showing that a maximal violation of BI in entangled neutral kaons is not possible in the presence of CP violation. Similar applications can be made in other examples as well, such as for entangled coherent states [3]. It may also be noted that in an experiment using orthogonal entangled states, any finite imprecision in the preparation of the entanglement can result in a departure from the required orthogonality. In order to analyse how such a deviation from the orthogonality in the prepared entangled state affects the relevant experimental result, the results presented in our paper could be useful.

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